

# CELLULAR COALGEBRAS OVER THE BARRATT-ECCLES OPERAD I.

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**ABSTRACT.** This paper considers a class of coalgebras over the Barratt-Eccles operad and shows that they classify  $\mathbb{Z}$ -completions of pointed, reduced simplicial sets. As a consequence, they encapsulate the homotopy types of nilpotent simplicial sets. This result is a direct generalization of Quillen's result characterizing rational homotopy types via cocommutative coalgebras.

## 1. INTRODUCTION

In [10] the author constructed coalgebra-structures over the Barratt-Eccles operad on the integral chain-complex of a pointed, reduced simplicial set. These coalgebra structures suffice to compute all Steenrod operations (among other things). The present paper shows that such coalgebras have an algebraic property called cellularity (see definition 3.8). Cellular coalgebras are shown to have an analogue of the Hurewicz map (corollary 3.11) that allows us to construct their “ $\mathbb{Z}$ -completion”. In the case that a cellular coalgebra is topologically realizable, properties of the space’s  $\mathbb{Z}$ -completion can be derived from that of the coalgebra.

The main technical result, theorem 3.10, implies that cellular coalgebras have a “Hurewicz map” that precisely corresponds to the topological Hurewicz map when a cellular coalgebra is topologically realizable.

We use this to derive a cosimplicial resolution of a cellular coalgebra and show that it is weakly equivalent to the  $\mathbb{Z}^\bullet$ -resolution when the coalgebra is derived from a topological space.

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Our final result is a kind of generalization of Quillen's main result (in [8]) characterizing rational homotopy types via commutative coalgebras:

*Corollary 3.22: If  $X$  and  $Y$  are pointed nilpotent reduced simplicial sets, then  $X$  is weakly equivalent to  $Y$  if and only if there exists a morphism of cellular coalgebras*

$$f: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$$

*that is an integral homology equivalence.*

An earlier somewhat similar statement was published by Smirnov in [9], but his proof was unclear and he used an operad that was uncountably generated in all dimensions. Smirnov's proof was so unclear that several people known to the author believed the result to be untrue.

The present paper's proof is a straightforward application of simplicial resolutions — involving the operad used to compute Steenrod operations.

Our result reduces homotopy theoretic questions to purely algebraic ones related to well-known invariants like the operation of the unstable Steenrod algebra.

## 2. DEFINITIONS AND ASSUMPTIONS

We will denote the closed symmetric monoidal category of  $\mathbb{Z}$ -free  $\mathbb{Z}$ -chain-complexes concentrated in positive dimensions by  $\mathbf{Ch}_0$ .

We make extensive use of the Koszul Convention (see [6]) regarding signs in homological calculations:

**Definition 2.1.** If  $f: C_1 \rightarrow D_1$ ,  $g: C_2 \rightarrow D_2$  are maps, and  $a \otimes b \in C_1 \otimes C_2$  (where  $a$  is a homogeneous element), then  $(f \otimes g)(a \otimes b)$  is defined to be  $(-1)^{\deg(g) \cdot \deg(a)} f(a) \otimes g(b)$ .

*Remark 2.2.* If  $f_i$ ,  $g_i$  are maps, it isn't hard to verify that the Koszul convention implies that  $(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2) \cdot \deg(g_1)} (f_1 \circ f_2 \otimes g_1 \circ g_2)$ .

**Definition 2.3.** Given chain-complexes  $A, B \in \mathbf{Ch}_0$  define

$$\mathrm{Hom}_{\mathbb{Z}}(A, B)$$

to be the chain-complex of graded  $\mathbb{Z}$ -morphisms where the degree of an element  $x \in \mathrm{Hom}_{\mathbb{Z}}(A, B)$  is its degree as a map and with differential

$$\partial f = f \circ \partial_A - (-1)^{\deg f} \partial_B \circ f$$

As a  $\mathbb{Z}$ -module  $\text{Hom}_{\mathbb{Z}}(A, B)_k = \prod_j \text{Hom}_{\mathbb{Z}}(A_j, B_{j+k})$ .

*Remark.* Given  $A, B \in \mathbf{Ch}^{S_n}$ , we can define  $\text{Hom}_{\mathbb{Z}S_n}(A, B)$  in a corresponding way.

**Definition 2.4.** Let  $\sigma \in S_n$  be an element of the symmetric group and let  $\{k_1, \dots, k_n\}$  be  $n$  nonnegative integers with  $K = \sum_{i=1}^n k_i$ . Then  $T_{k_1, \dots, k_n}(\sigma)$  is defined to be the element  $\tau \in S_K$  that permutes the  $n$  blocks

$$(1, \dots, k_1), (k_1 + 1, \dots, k_1 + k_2) \dots (k - k_{n-1}, \dots, k)$$

as  $\sigma$  permutes the set  $\{1, \dots, n\}$ .

*Remark 2.5.* Note that it is possible for one of the  $k$ 's to be 0, in which case the corresponding block is empty.

**Definition 2.6.** If  $G$  is a discrete group, let  $\mathbf{Ch}_0^G$  denote the category of chain-complexes equipped with a right  $G$ -action. This is again a closed symmetric monoidal category and the forgetful functor  $\mathbf{Ch}_0^G \rightarrow \mathbf{Ch}_0$  has a left adjoint,  $(-)[G]$ . This applies to the symmetric groups,  $S_n$ , where we regard  $S_1$  and  $S_0$  as the trivial group. The *category of collections* is defined to be the product

$$\text{Coll}(\mathbf{Ch}_0) = \prod_{n \geq 0} \mathbf{Ch}_0^{S_n}$$

Its objects are written  $\mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0}$ . Each collection induces an endofunctor (also denoted  $\mathcal{V}$ )  $\mathcal{V}: \mathbf{Ch}_0 \rightarrow \mathbf{Ch}_0$

$$\mathcal{V}(X) = \bigoplus_{n \geq 0} \mathcal{V}(n) \otimes_{\mathbb{Z}S_n} X^{\otimes n}$$

where  $X^{\otimes n} = X \otimes \cdots \otimes X$  and  $S_n$  acts on  $X^{\otimes n}$  by permuting factors. This endofunctor is a *monad* if the defining collection has the structure of an *operad*, which means that  $\mathcal{V}$  has a unit  $\eta: \mathbb{Z} \rightarrow \mathcal{V}(1)$  and structure maps

$$\gamma_{k_1, \dots, k_n}: \mathcal{V}(n) \otimes \mathcal{V}(k_1) \otimes \cdots \otimes \mathcal{V}(k_n) \rightarrow \mathcal{V}(k_1 + \cdots + k_n)$$

satisfying well-known equivariance, associativity, and unit conditions — see [11], [7].

We will call the operad  $\mathcal{V} = \{\mathcal{V}(n)\}$   $\Sigma$ -cofibrant if  $\mathcal{V}(n)$  is  $\mathbb{Z}S_n$ -projective for all  $n \geq 0$ .

*Remark.* The operads we consider here correspond to *symmetric* operads in [11].

The term “unital operad” is used in different ways by different authors. We use it in the sense of Kriz and May in [7],

meaning the operad has a 0-component that acts like an arity-lowering augmentation under compositions. Here  $\mathcal{V}(0) = \mathbb{Z}$ .

The term  $\Sigma$ -cofibrant first appeared in [3].

A simple example of an operad is:

**Example 2.7.** For each  $n \geq 0$ ,  $X$ ,  $C(n) = \mathbb{Z}S_n$ , with structure-map induced by

$$\gamma_{\alpha_1, \dots, \alpha_n}: S_n \times S_{\alpha_1} \times \cdots \times S_{\alpha_n} \rightarrow S_{\alpha_1 + \cdots + \alpha_n}$$

defined by regarding each of the  $S_{\alpha_i}$  as permuting elements within the subsequence  $\{\alpha_1 + \cdots + \alpha_{i-1} + 1, \dots, \alpha_1 + \cdots + \alpha_i\}$  of the sequence  $\{1, \dots, \alpha_1 + \cdots + \alpha_n\}$  and making  $S_n$  permute these  $n$ -blocks. This operad is denoted  $\mathfrak{S}_0$ . In other notation, its  $n^{\text{th}}$  component is the *symmetric group-ring*  $\mathbb{Z}S_n$ . See [10] for explicit formulas.

For the purposes of this paper, the main example of an operad is

**Definition 2.8.** Given any  $C \in \mathbf{Ch}_0$ , the associated *coendomorphism operad*,  $\text{CoEnd}(C)$  is defined by

$$\text{CoEnd}(C)(n) = \text{Hom}_{\mathbb{Z}}(C, C^{\otimes n})$$

Its structure map

$$\gamma_{\alpha_1, \dots, \alpha_n}:$$

$$\begin{aligned} \text{Hom}_{\mathbb{Z}}(C, C^{\otimes n}) \otimes \text{Hom}_{\mathbb{Z}}(C, C^{\otimes \alpha_1}) \otimes \cdots \otimes \text{Hom}_{\mathbb{Z}}(C, C^{\otimes \alpha_n}) \rightarrow \\ \text{Hom}_{\mathbb{Z}}(C, C^{\otimes \alpha_1 + \cdots + \alpha_n}) \end{aligned}$$

simply composes a map in  $\text{Hom}_{\mathbb{Z}}(C, C^{\otimes n})$  with maps of each of the  $n$  factors of  $C$ .

This is a non-unital operad, but if  $C \in \mathbf{Ch}_0$  has an augmentation map  $\varepsilon: C \rightarrow \mathbb{Z}$  then we can regard  $\varepsilon$  as the only element of  $\text{Hom}_{\mathbb{Z}}(C, C^{\otimes n}) = \text{Hom}_{\mathbb{Z}}(C, C^{\otimes 0}) = \text{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$ .

Morphisms of operads are defined in the obvious way:

**Definition 2.9.** Given two operads  $\mathcal{V}$  and  $\mathcal{W}$ , a *morphism*

$$f: \mathcal{V} \rightarrow \mathcal{W}$$

is a sequence of chain-maps

$$f_i: \mathcal{V}_i \rightarrow \mathcal{W}_i$$

commuting with all the diagrams in 2.6.

Verification that this satisfies the required identities is left to the reader as an exercise.

**Definition 2.10.** Let  $\mathbb{S}$  denote the Barratt-Eccles operad with components  $K(S_n, 1)$  — the bar resolutions of  $\mathbb{Z}$  over  $\mathbb{Z}S_n$  for all  $n > 0$ . See [10] for formulas for the composition-operations. Coalgebras over  $\mathbb{S}$  are chain-complexes equipped with a coassociative coproduct and Steenrod operations for all primes (see [10]).

*Remark.* The operad  $\mathbb{S}$  was first described in [2].

**Definition 2.11.** A chain-complex  $C$  is a *coalgebra over the operad  $\mathcal{V}$*  if there exists a morphism of operads

$$\mathcal{V} \rightarrow \text{CoEnd}(C)$$

The structure of a coalgebra over an operad can be described in several equivalent ways:

- (1)  $f_n: \mathcal{V}(n) \otimes C \rightarrow C^{\otimes n}$
- (2)  $g: C \rightarrow \prod_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(\mathcal{V}(n), C^{\otimes n})$

where both satisfy identities that describe how composites of these maps are compatible with the operad-structure.

### 2.1. Types of coalgebras.

**Example 2.12.** Coassociative coalgebras are precisely the coalgebras over  $\mathbb{S}_0$  (see 2.7).

**Definition 2.13.** Cocommut is an operad defined to have one basis element  $\{b_i\}$  for each integer  $i \geq 0$ . Here the rank of  $b_i$  is  $i$  and the degree is 0 and the these elements satisfy the composition-law:  $\gamma(b_n \otimes b_{k_1} \otimes \cdots \otimes b_{k_n}) = b_K$ , where  $K = \sum_{i=1}^n k_i$ . The differential of this operad is identically zero. The symmetric-group actions are trivial.

**Example 2.14.** Coassociative, commutative coalgebras are the coalgebras over Cocommut.

We will sometimes want to focus on a particular class of  $\mathcal{V}$ -coalgebras: the *pointed, irreducible coalgebras*. We define this concept in a way that extends the conventional definition in [12]:

**Definition 2.15.** Given a coalgebra over a unital operad  $\mathcal{V}$  with adjoint structure-maps

$$a_n: C \rightarrow \text{Hom}_{\mathbb{Z}S_n}(\mathcal{V}(n), C^{\otimes n})$$

for  $n \geq 0$ , an element  $c \in C$  is called *group-like* if  $a_n(c) = \text{Hom}_{\mathbb{Z}}(\epsilon_n, 1)(1 \mapsto c^{\otimes n})$  for all  $n > 0$ . Here  $c^{\otimes n} \in C^{\otimes n}$  is the  $n$ -fold  $\mathbb{Z}$ -tensor product and  $\epsilon_n: \mathcal{V}(n) \rightarrow \mathcal{V}(0) = \mathbb{Z}$  is the augmentation (which is  $n$ -fold composition with  $\mathcal{V}(0)$ ).

A coalgebra  $C$  over an operad  $\mathcal{V}$  is called *pointed* if it has a *unique* group-like element (denoted 1), and *pointed irreducible* if the intersection of any two sub-coalgebras contains this unique group-like element.

*Remark.* Note that a group-like element generates a sub  $\mathcal{V}$ -coalgebra of  $C$  and must lie in dimension 0.

Although seemingly contrived, this condition arises in “nature”: The chain-complex of a pointed, reduced simplicial set is naturally a pointed irreducible coalgebra over the Barratt-Eccles operad,  $\mathfrak{S} = \{C(K(S_n, 1))\}$  (see [10]).

**Proposition 2.16.** *Let  $D$  be a pointed, irreducible coalgebra over an operad  $\mathcal{V}$ . Then the augmentation map*

$$\varepsilon: D \rightarrow \mathbb{Z}$$

*is naturally split and any morphism of pointed, irreducible coalgebras*

$$f: D_1 \rightarrow D_2$$

*is of the form*

$$1 \oplus \bar{f}: D_1 = \mathbb{Z} \oplus \ker \varepsilon_{D_1} \rightarrow D_2 = \mathbb{Z} \oplus \ker \varepsilon_{D_2}$$

*where  $\varepsilon_i: D_i \rightarrow \mathbb{Z}$ ,  $i = 1, 2$  are the augmentations.*

*Proof.* The definition (2.15) of the sub-coalgebra  $\mathbb{Z} \cdot 1 \subseteq D_i$  is stated in an invariant way, so that any coalgebra morphism must preserve it. Any morphism must also preserve augmentations because the augmentation is the 0<sup>th</sup>-order structure-map. Consequently,  $f$  must map  $\ker \varepsilon_{D_1}$  to  $\ker \varepsilon_{D_2}$ . The conclusion follows.  $\square$

**Definition 2.17.** We denote the *category* of pointed irreducible coalgebras over  $\mathfrak{S}$  by  $\mathcal{I}_0$ . Every such coalgebra,  $C$ , comes equipped with a canonical augmentation

$$\varepsilon: C \rightarrow \mathbb{Z}$$

so the *terminal object* is  $\mathbb{Z}$ . If  $\mathcal{V}$  is not unital, the terminal object in this category is 0, the null coalgebra.

The category of *pointed irreducible coalgebras* over  $\mathfrak{S}$  is denoted  $\mathcal{I}_0$ . Its terminal object is the coalgebra whose underlying chain complex is  $\mathbb{Z}$  concentrated in dimension 0.

We also need:

**Definition 2.18.** If  $A \in \mathcal{C} = \mathcal{I}_0$  or  $\mathcal{S}_0$ , then  $[A]$  denotes the underlying chain-complex in  $\text{Ch}_0$  of

$$\ker A \rightarrow t$$

where  $t$  denotes the terminal object in  $\mathcal{C}$  — see definition 2.17. We will call  $[*]$  the *forgetful functor* from  $\mathcal{C}$  to  $\text{Ch}_0$ .

We can define a concept dual to that of a free algebra generated by a set:

**Definition 2.19.** Let  $D$  be a coalgebra over an operad  $\mathfrak{S}$ , equipped with a  $\text{Ch}_0$ -morphism  $\varepsilon: [D] \rightarrow E$ , where  $E \in \text{Ch}_0$ . Then  $D$  is called the *cofree coalgebra over  $\mathfrak{S}$  cogenerated by  $\varepsilon$*  if any morphism in  $\text{Ch}_0$

$$f: [C] \rightarrow E$$

where  $C \in \mathcal{S}_0$ , induces a *unique* morphism in  $\mathcal{S}_0$

$$\alpha_f: C \rightarrow D$$

that makes the diagram

$$\begin{array}{ccc} [C] & \xrightarrow{[\alpha_f]} & [D] \\ & \searrow f & \downarrow \varepsilon \\ & & E \end{array}$$

commute. Here  $\alpha_f$  is called the *classifying map* of  $f$ . If  $C \in \mathcal{S}_0$  then

$$\alpha_f: C \rightarrow L_{\mathfrak{S}}[C]$$

will be called the *classifying map of  $C$* .

This universal property of cofree coalgebras implies that they are unique up to isomorphism if they exist.

The paper [11] explicitly constructs cofree coalgebras for many operads, including  $\mathfrak{S}$ :

- $L_{\mathfrak{S}}C$  is the general cofree coalgebra over the operad  $\mathfrak{S}$  — here,  $C$ , is a chain-complex that is not necessarily concentrated in nonnegative dimensions.
- $P_{\mathfrak{S}}C$  is the *pointed irreducible* cofree coalgebra for  $C$  (see definition 2.15 and [11, section 4.2]).

In all cases, definition 2.19 implies the existence of an adjunction

$$(2.1) \quad [*]: \mathbf{Ch}_0 \leftrightarrows \mathcal{I}_0: P_{\mathfrak{S}*}$$

where  $[*]: \mathcal{I}_0 \rightarrow \mathbf{Ch}_0$  is the forgetful functor.

### 3. CELLULAR COAGEBRAS

We recall the following, from [10, chapter 2, proposition 4.3]:

**Definition 3.1.** The functor

$$\mathcal{C}(*): \mathbf{S} \rightarrow \mathcal{S}_0$$

from simplicial sets to  $\mathfrak{S}$ -coalgebras sends a simplicial set to its chain-complex equipped with an  $\mathfrak{S}$ -coalgebra structure defined via acyclic models on the simplices.

A  $\mathfrak{S}$ -coalgebra,  $D$ , will be said to be *strongly realizable* if  $D \cong \mathcal{C}(X)$  for some pointed, reduced simplicial set  $X$ .

*Remark.* The  $\mathfrak{S}$ -coalgebra structure coincides with that used to define Steenrod operations. More accurately, [10, chapter 2] shows that the “higher coproducts” used to define these operations are part of an  $\mathfrak{S}$ -coalgebra structure.

We will also define a complementary functor

**Definition 3.2.** Define a functor

$$\hom(\star, *): \mathcal{I}_0 \rightarrow \mathbf{S}_0$$

as follows:

If  $C \in \mathcal{I}_0$ , define the  $n$ -simplices of  $\hom(\star, C)$  to be the  $\mathfrak{S}$ -coalgebra morphisms

$$\mathcal{C}(\Delta^n) \rightarrow C$$

where  $\Delta^n$  is the standard  $n$ -simplex. Face-operators are defined by inclusion of faces and degeneracies in a corresponding fashion.

*Remark.* The normalized chain complex satisfies:

$$(3.1) \quad C(\Delta^n)_k = \bigoplus_{n \rightarrow k} \mathbb{Z} = \mathbb{Z}^{\binom{n+1}{n-k}}$$

where  $n \rightarrow k$  runs over all ordered surjections

$$[0, \dots, n] \rightarrow [0, \dots, k]$$

This is similar to the definition of the Dold-Kan functor

$$\Gamma: \mathbf{Ch}_0 \rightarrow \mathbf{sAb}$$

to the category of simplicial abelian groups (see [5, chapter III, section 2]), the essential difference being that  $\text{hom}(\star, *)$  takes  $\mathfrak{S}$ -coalgebra structures into account.

**Lemma 3.3.** *The functors  $\mathcal{C}(*)$  and  $\text{hom}(\star, *)$  define an adjunction*

$$\mathcal{C}(*) : \mathcal{I}_0 \leftrightarrows \mathbf{S}_0 : \text{hom}(\star, *)$$

This gives rise to natural transformations (in the appropriate categories)

$$\begin{aligned} u_X : X &\rightarrow \text{hom}(\star, \mathcal{C}(X)) \\ w_D : \mathcal{C}(\text{hom}(\star, D)) &\rightarrow D \end{aligned}$$

*Proof.* We must show that, for any  $X \in \mathbf{S}_0$  and  $D \in \mathcal{I}_{\text{cell}}$  we have a bijection

$$\text{hom}_{\mathcal{I}_{\text{cell}}}(\mathcal{C}(X), D) = \text{hom}_{\mathbf{S}_0}(X, \text{hom}(\star, D))$$

If we start on the left side, it is not hard to see that every morphism  $\mathcal{C}(X) \rightarrow D$  can be regarded as a collection of morphisms  $\mathcal{C}(\Delta^k) \rightarrow D$  for each simplex  $\Delta^k$  of  $X$ . This also defines a mapping  $X \rightarrow \text{hom}(\star, D)$ . The converse argument is also straightforward.  $\square$

**Proposition 3.4.** *If  $X$  is a pointed, reduced simplicial set the adjunction in lemma 3.3 implies the existence of a coalgebra morphism*

$$\mathcal{C}(u_X) : \mathcal{C}(X) \rightarrow \mathcal{C}(\text{hom}(\star, \mathcal{C}(X)))$$

such that  $w_D \circ \mathcal{C}(u_X) = 1 : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$ . Given pointed, reduced simplicial set  $Y$ , and a morphism of  $\mathfrak{S}$ -coalgebras,

$$f : \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$$

the diagram

$$\begin{array}{ccc} \mathcal{C}(\text{hom}(\star, \mathcal{C}(X))) & \xrightarrow{\text{hom}(\star, f)} & \mathcal{C}(\text{hom}(\star, \mathcal{C}(Y))) \\ \mathcal{C}(u_X) \uparrow & & \uparrow \mathcal{C}(u_Y) \\ \mathcal{C}(X) & \xrightarrow{f} & \mathcal{C}(Y) \end{array}$$

commute.

**Remark.** If  $D \in \mathcal{I}_0$ , the existence of a morphism  $D \rightarrow \mathcal{C}(\text{hom}(\star, D))$  splitting the canonical morphism  $\mathcal{C}(\text{hom}(\star, D)) \rightarrow D$  is a necessary condition for  $D$  to be strongly realizable.

The (slightly) interesting thing about this diagram is that its vertical maps depend on the topological realizability of  $\mathcal{C}(X)$  and  $\mathcal{C}(Y)$  but the horizontal maps need not be topologically realizable.

*Proof.* If  $x \in X$  is a simplex and  $f(x) = \sum a_i y_i$  for simplices  $y_i \in Y$ , then a simple diagram-chase confirms the conclusion.  $\square$

**Definition 3.5.** If  $C$  is a chain-complex with augmentation  $\epsilon: C \rightarrow \mathbb{Z}$ , define  $\tilde{\Gamma}C$  to be the  $\Gamma \ker \epsilon$ , where  $\Gamma^*$  is the Dold-Kan functor. If  $C \in \text{Ch}_0$ , set  $\tilde{\Gamma}C = GC$ .

*Remark.* This is just a pointed version of the Dold-Kan functor.

The following is probably well-known, but we will use it heavily:

**Lemma 3.6.** *If  $X \in S_0$  is a pointed, reduced simplicial set, there is a natural trivial fibration*

$$\gamma_X: \tilde{\Gamma}C(X) \rightarrow \tilde{\mathbb{Z}}X$$

*and a natural trivial cofibration*

$$\iota_X: \tilde{\mathbb{Z}}X \rightarrow \tilde{\Gamma}C(X)$$

*of simplicial abelian groups, where  $C(X)$  is the (integral) chain-complex and  $\tilde{\mathbb{Z}}X = \mathbb{Z}X/\mathbb{Z}^*$  — the pointed version of the free abelian group functor. In addition,  $\gamma_X \circ \iota_X = 1: \tilde{\mathbb{Z}}X \rightarrow \tilde{\mathbb{Z}}X$ .*

*Remark.* It is well-known that  $\tilde{\Gamma}$  defines an equivalence of categories where  $sAB_0$  is the category of pointed reduced simplicial abelian groups — see [5, chapter III].

*Proof.* Consider the surjective homology equivalence

$$\mathbb{Z}X = C(X) \rightarrow C(X)/DC(X) = NC(X)$$

where  $DC(X)$  is the subcomplex generated by degenerates and  $NC(X)$  is the normalized chain-complex. Now, regard  $\mathbb{Z}X$  as a chain-complex and take  $\tilde{\Gamma}(\cdot)$  of this surjection. We get

$$\tilde{\Gamma}C(X) \rightarrow \tilde{\Gamma}NC(X) = \tilde{\mathbb{Z}}X$$

by the Dold-Kan correspondence — see [5, chapter III], corollary 2.3, theorem 2.5, and corollary 2.12. The second statement follows by similar reasoning, using the split inclusion

$$NC(X) \rightarrow C(X)$$

Both maps clearly preserve the abelian group structure.  $\square$

Recall that there is an adjunction

$$(3.2) \quad C(*) : \mathbf{Ch}_0 \leftrightarrows \mathbf{S}_0 : \Gamma *$$

where  $C(*)$  is the integral chain-complex functor (see [5, chapter III]).

**Definition 3.7.** If  $C \in \mathbf{Ch}_0$ , define  $p_C : [\mathcal{C}(\tilde{\Gamma}C)] \rightarrow C$  to be the chain-map that corresponds to the *identity map*  $1 : \tilde{\Gamma}D \rightarrow \tilde{\Gamma}D$  in 3.2, i.e.

$$\hom_{\mathbf{Ch}_0}(C(\tilde{\Gamma}D), D) = \hom_{\mathbf{S}_0}(\tilde{\Gamma}D, \tilde{\Gamma}D)$$

The chain-map  $p_C$  induces a canonical coalgebra-morphism (via the adjunction in equation 2.1):

$$a_C : \mathcal{C}(\tilde{\Gamma}C) \rightarrow P_{\mathfrak{S}}(C)$$

where  $P_{\mathfrak{S}}(C)$  is the pointed, irreducible cofree coalgebra constructed in [11].

*Remark.* The *other* map defined by the adjunction in 3.2 is

$$X \rightarrow \tilde{\Gamma}[\mathcal{C}(X)]$$

which is essentially the Hurewicz map.

The maps  $p_C$  allow us to define *cellular coalgebras*:

**Definition 3.8.** If  $a_D : \mathcal{C}(\tilde{\Gamma}[D]) \rightarrow P_{\mathfrak{S}}[D]$  is the map in definition 3.7, define

$$L_{\text{cell}}[D] = \text{im } \mathcal{C}(\tilde{\Gamma}[D]) \subset P_{\mathfrak{S}}[D]$$

A  $\mathbb{Z}$ -free pointed, irreducible  $\mathfrak{S}$ -coalgebra,  $D$ , will be called *cellular* if its classifying map (see definition 2.19 and equation 2.1) satisfies

$$\beta_D : D \rightarrow L_{\text{cell}}D \subset P_{\mathfrak{S}}[D]$$

The category of cellular coalgebras will be denoted  $\mathcal{S}_{\text{cell}}$ . *Morphisms* of cellular coalgebras will simply be  $\mathfrak{S}$ -coalgebra morphisms.

*Remark.* Cellular coalgebras are necessarily concentrated in nonnegative dimensions.

Definition 2.19 and the *uniqueness* of the morphism  $\alpha_f$  imply that:

**Theorem 3.9** (Rigidity Theorem). *Let  $C \in \mathcal{S}_{\text{cell}}$ , let  $D \in \mathbf{Ch}_0$  and let  $f: [C] \rightarrow D$  be any chain-map. Then there exists a unique  $\mathfrak{S}$ -coalgebra morphism  $\hat{f}: C \rightarrow L_{\text{cell}}D$  that makes the diagram*

$$\begin{array}{ccc} [C] & \xrightarrow{[\hat{f}]} & L_{\text{cell}}D \\ & \searrow f & \downarrow p_D \\ & & D \end{array}$$

*commute. It follows that we have an adjunction*

$$[*]: \mathbf{Ch}_0 \leftrightarrows \mathcal{S}_{\text{cell}}: L_{\text{cell}}*$$

*where  $[*]: \mathcal{S}_{\text{cell}} \rightarrow \mathbf{Ch}_0$  is the forgetful functor (compare this with equation 2.1).*

*Proof.* Definition 2.19 implies that there exists a unique map  $\hat{f}: C \rightarrow P_{\mathfrak{S}}D$  making the diagram

$$\begin{array}{ccc} [C] & \xrightarrow{[\hat{f}]} & [P_{\mathfrak{S}}D] \\ & \searrow f & \downarrow p_D \\ & & D \end{array}$$

commute. Here  $P_{\mathfrak{S}}D$  is the pointed, irreducible cofree coalgebra cogenerated by  $D$  (see [11]). The cellularity of  $C$  implies that the unique map  $C \rightarrow P_{\mathfrak{S}}[C]$  has its image in  $L_{\text{cell}}[C] \subset P_{\mathfrak{S}}[C]$  and conclusion follows from the diagram

$$\begin{array}{ccc} L_{\text{cell}}[C] & \xrightarrow{a_{[C]}} & P_{\mathfrak{S}}[C] \\ \mathcal{C}(\tilde{\Gamma}f) \downarrow & & \downarrow P_{\mathfrak{S}}f \\ L_{\text{cell}}D & \xrightarrow{a_D} & P_{\mathfrak{S}}D \end{array}$$

which shows that there is a coalgebra-morphism  $C \rightarrow \tilde{\Gamma}D$  covering  $f$ . The conclusion follows from the fact that its composite with the inclusion  $L_{\text{cell}}D \rightarrow P_{\mathfrak{S}}D$  is *unique*.  $\square$

One of the key ideas in this paper is:

**Theorem 3.10** (Injectivity Theorem). *Under the conditions of definition 3.7, if  $X$  is a pointed, reduced simplicial set, the composite*

$$\mathcal{C}(\tilde{\mathbb{Z}}X) \xrightarrow{\mathcal{C}(\iota_X)} \mathcal{C}(\tilde{\Gamma}C) \xrightarrow{a_C} P_{\mathfrak{S}}C$$

*is injective, where  $C = [\mathcal{C}(X)]$  and  $\iota_X: \tilde{\mathbb{Z}}X \rightarrow \tilde{\Gamma}X$  is defined in lemma 3.6*

*Proof.* See appendix A. □

This immediately implies

**Corollary 3.11.** *If  $X$  is a pointed, reduced simplicial set, then*

$$L_{\text{cell}}[\mathcal{C}(X)] = \mathcal{C}(\tilde{\mathbb{Z}}X)$$

*It follows that  $\mathcal{C}(X)$  is cellular and that the classifying map*

$$\beta_{\mathcal{C}(X)}: \mathcal{C}(X) \rightarrow L_{\text{cell}}[\mathcal{C}(X)] = \mathcal{C}(\tilde{\mathbb{Z}}X)$$

*is the chain-map induced by the Hurewicz map.*

*If  $\varepsilon: [\mathcal{C}(\tilde{\mathbb{Z}}X)] = [L_{\text{cell}}[\mathcal{C}(X)]] \rightarrow [\mathcal{C}(X)]$  is the cogeneration map, then*

$$\varepsilon \left( \sum_i a_i \left[ \sum_j b_{i,j} \cdot \sigma_{i,j} \right] \right) = \sum_{i,j} a_i b_{i,j} \sigma_{i,j}$$

*where the  $\sigma_{i,j} \in X$  are generators of  $[\mathcal{C}(X)]$ ,  $[\sum_j b_{i,j} \cdot \sigma_{i,j}] \in \tilde{\mathbb{Z}}X$  are generators of  $[\mathcal{C}(\tilde{\mathbb{Z}}X)]$ , and the  $a_i, b_{i,j} \in \mathbb{Z}$ . It follows that  $L_{\text{cell}}\varepsilon: L_{\text{cell}}[L_{\text{cell}}[\mathcal{C}(X)]] = [\mathcal{C}(\tilde{\mathbb{Z}}^2X)] \rightarrow [\mathcal{C}(\tilde{\mathbb{Z}}X)]$  is given by*

$$(3.3) \quad L_{\text{cell}}\varepsilon \left( \left[ \sum_i a_i \left[ \sum_j b_{i,j} \cdot \sigma_{i,j} \right] \right] \right) = \left[ \sum_{i,j} a_i b_{i,j} \cdot \sigma_{i,j} \right]$$

*Remark.* Among other things, this and definition 2.19 (or theorem 3.9) imply that the geometrically-relevant Hurewicz map

$$\mathcal{C}(X) \rightarrow \mathcal{C}(\tilde{\mathbb{Z}}X) \subset P_{\mathfrak{S}}[\mathcal{C}(X)]$$

is uniquely determined by the coalgebra structure of  $\mathcal{C}(X)$ .

Proposition 2.20 in chapter III of [5] implies that  $L_{\text{cell}}D$  is (unnaturally) chain-homotopy equivalent to

$$\mathcal{C}\left(\prod_{j=1}^{\infty} K(H_j(D), j)\right)$$

where  $H_j(D)$  is the  $j^{\text{th}}$  homology group.

It is well-known that the chain-complex of a space like  $X$  determines the homotopy type of  $\mathbb{Z}X$  (see [5, chapter III, section 2]). It is natural to ask what additional information the coalgebra structure provides. This proposition gives the answer: it determines the chain-level effect of the *Hurewicz map*. But these two data-points (i.e., the chain-complex and the chain-map induced by the Hurewicz map) suffice to define  $\mathbb{Z}^\bullet X$  — the cosimplicial space used to construct Bousfield and Kan's  $\mathbb{Z}$ -completion,  $\mathbb{Z}_\infty X$  (see [1]).

*Proof.* Definition 2.19 and the fact that  $P_{\mathfrak{S}}[\mathcal{C}(X)]$  is cofree implies that the diagram

$$\begin{array}{ccc} \mathcal{C}(\tilde{\Gamma}[\mathcal{C}(X)]) & \xrightarrow{\mathcal{C}(\gamma_X)} & \mathcal{C}(\tilde{\mathbb{Z}}X) \\ & \searrow a_{\mathcal{C}(X)} & \downarrow a_{\mathcal{C}(\tilde{\mathbb{Z}}X)} \\ & & P_{\mathfrak{S}}[\mathcal{C}(X)] \end{array}$$

commutes. Since  $\mathcal{C}(\gamma_X)$  is surjective and  $a_{\mathcal{C}(\tilde{\mathbb{Z}}X)}$  is injective (by theorem 3.10) it follows that the image of  $a_{\mathcal{C}(X)}$  is isomorphic to  $\mathcal{C}(\tilde{\mathbb{Z}}X)$ . The Hurewicz map  $h: X \rightarrow \tilde{\mathbb{Z}}X$  induces an  $\mathfrak{S}$ -coalgebra morphism  $\mathcal{C}(X) \rightarrow \mathcal{C}(\tilde{\mathbb{Z}}X)$  making the composite

$$(3.4) \quad \mathcal{C}(X) \xrightarrow{\mathcal{C}(h)} \mathcal{C}(\tilde{\mathbb{Z}}X) \xrightarrow{\mathcal{C}(\iota_X)} \mathcal{C}(\tilde{\Gamma}[\mathcal{C}(X)]) \rightarrow L_{\text{cell}}[\mathcal{C}(X)]$$

an  $\mathfrak{S}$ -coalgebra morphism. Here  $\iota_X: \tilde{\mathbb{Z}}X \rightarrow \tilde{\Gamma}[\mathcal{C}(X)]$  is defined in lemma 3.6. Since the composite of this with the inclusion with the inclusion  $\mathcal{C}(\tilde{\Gamma}[\mathcal{C}(X)]) \hookrightarrow P_{\mathfrak{S}}[\mathcal{C}(X)]$  is *unique* (see definition 2.19 and equation 2.1), it follows that the composite in 3.4 is the classifying map of  $\mathcal{C}(X)$ , which must be cellular.

The assertion regarding  $\varepsilon$  is clear.

We can also conclude that: □

**Corollary 3.12.** *If  $C \in \mathbf{Ch}_0$  then  $\hom(\star, L_{\text{cell}}C) = \tilde{\Gamma}C$ .*

*Proof.* The rigidity theorem (3.9) implies that

$$\hom(\star, L_{\text{cell}}C) = \{\hom_{\mathcal{S}_0}(\mathcal{C}(\Delta^i), L_{\text{cell}}C)\} = \{\hom_{\mathbf{Ch}_0}(C(\Delta^i), C)\}$$

where  $i$  runs over simplices of all dimensions. But equation 3.1 implies that this is just the definition of  $\tilde{\Gamma}C$  (see [5, chapter III]). □

**Definition 3.13.** Let:

- (1)  $\text{coS}$  denote the category of cosimplicial, pointed reduced simplicial sets,
- (2)  $\text{co}\mathcal{S}_{\text{cell}}$  denote the category of cosimplicial cellular coalgebras,
- (3)  $\text{coCh}_0$  denote the category of cosimplicial chain-complexes
- (4)  $\text{cohom}(\star, *): \text{co}\mathcal{S}_{\text{cell}} \rightarrow \text{coS}$  be the result of applying  $\hom(\star, *)$  in each codimension

**Proposition 3.14.** *The list  $(L_{\text{cell}}[*], \beta_*, v_*)$  constitutes a triple (or monad — see [13, section 8.6]) on  $\mathcal{S}_{\text{cell}}$  where, for  $C \in \mathcal{S}_{\text{cell}}$ :*

- (1)  $\beta_C: C \rightarrow L_{\text{cell}}[C]$  is the classifying map,

(2)  $v_C = L_{\text{cell}}\epsilon: L_{\text{cell}}([L_{\text{cell}}C]) \rightarrow L_{\text{cell}}C$ , where  $\varepsilon: L_{\text{cell}}C \rightarrow C$  is the restriction of the cogeneration map  $[P_{\mathfrak{S}}C] \rightarrow C$  (see definition 2.19).

Consequently, it defines a functor

$$Q: \mathcal{S}_{\text{cell}} \rightarrow \mathbf{co}\mathcal{S}_{\text{cell}}$$

(see [13, section 8.6]).

Composing  $Q(*)$  with the functor  $\text{hom}(\star, *)$ , applied in each codimension, gives rise to a functor

$$\hat{Q}: \mathcal{S}_{\text{cell}} \rightarrow \mathbf{co}\mathcal{S}$$

to the category of pointed reduced cosimplicial sets.

*Remark.* The cogeneration-map

$$\varepsilon: [P_{\mathfrak{S}}C] \rightarrow C$$

is defined by the composite

$$P_{\mathfrak{S}}C \hookrightarrow \prod_{j=0}^{\infty} \text{Hom}_{\mathbb{Z}S_j}(\text{RS}_j, C^{\otimes j}) \rightarrow \text{Hom}_{\mathbb{Z}S_1}(\text{RS}_1, C) = C$$

see [11]. Here, we follow the convention that  $\text{Hom}_{\mathbb{Z}S_0}(\text{RS}_1, C^{\otimes 0}) = \mathbb{Z}$  and  $\text{Hom}_{\mathbb{Z}S_1}(\text{RS}_1, C) = C$ .

*Proof.* The diagram

$$\begin{array}{ccccc} L_{\text{cell}}[(L_{\text{cell}}[(L_{\text{cell}}[C])])] & \xlongequal{\quad} & L_{\text{cell}}[(L_{\text{cell}}[L_{\text{cell}}[C]])] & \xrightarrow{L_{\text{cell}}[v_c]} & L_{\text{cell}}[L_{\text{cell}}[C]] \\ v_{L_{\text{cell}}C} \downarrow & & & & \downarrow v_C \\ L_{\text{cell}}[L_{\text{cell}}[C]] & \xrightarrow{v_C} & & & \rightarrow L_{\text{cell}}[C] \end{array}$$

commutes by virtue of the Rigidity Theorem (3.9): in all composed maps,  $C$ , remains fixed so the composites are the unique coalgebra morphisms that lift the cogeneration map  $\varepsilon: L_{\text{cell}}C \rightarrow C$ . A similar argument shows that the diagram

$$\begin{array}{ccc} L_{\text{cell}}C & \xrightarrow{L_{\text{cell}}\beta_C} & L_{\text{cell}}[L_{\text{cell}}C] & \xleftarrow{\beta_{L_{\text{cell}}[C]}} & L_{\text{cell}}C \\ & \searrow & \downarrow v_C & \swarrow & \\ & & L_{\text{cell}}C & & \end{array}$$

commutes, verifying the identities that  $(L_{\text{cell}}*, \beta_*, v_*)$  must satisfy. The construction of  $QC$  is also standard — see [13, section 8.6] or [5, chapter VII, section 4]. The final statement is straightforward.  $\square$

In the case where our cellular coalgebra is of the form  $\mathcal{C}(X)$ , we can say more about  $Q_*$ :

**Lemma 3.15.** *If  $X$  is a pointed, reduced simplicial set, the cosimplicial cellular coalgebra  $Q\mathcal{C}(X)$  has levels  $Q^n\mathcal{C}(X) = \mathcal{C}(\tilde{\mathbb{Z}}^{n+1}X)$  with*

(1) *coface maps*

$$\delta^i = \mathcal{C}(\tilde{\mathbb{Z}}^i h \tilde{\mathbb{Z}}^{n-i+1}):$$

$$Q^n\mathcal{C}(X) = \mathcal{C}(\tilde{\mathbb{Z}}^{n+1}X) \rightarrow Q^{n+1}\mathcal{C}(X) = \mathcal{C}(\tilde{\mathbb{Z}}^{n+2}X)$$

for  $i = 0, \dots, n+1$ , where  $h: X \rightarrow \tilde{\mathbb{Z}}X$  is the Hurewicz map,  
and

(2) *codegeneracy maps*

$$s^i = \mathcal{C}(\tilde{\mathbb{Z}}^i \gamma \tilde{\mathbb{Z}}^{n-i}):$$

$$Q^{n+1}\mathcal{C}(X) = \mathcal{C}(\tilde{\mathbb{Z}}^{n+2}X) \rightarrow Q^n\mathcal{C}(X) = \mathcal{C}(\tilde{\mathbb{Z}}^{n+1}X)$$

for  $i = 0, \dots, n$ , where  $\gamma = L_{\text{cell}}\varepsilon$  in equation 3.3.

It follows that

$$Q^\bullet\mathcal{C}(X) = \mathcal{C}(\mathbb{Z}^\bullet X)$$

where  $\mathbb{Z}^\bullet X$  is the cosimplicial  $\mathbb{Z}$ -resolution of  $X$  defined in example 4.1 of [5, chapter VII, section 4].

*Remark.* The cosimplicial space,  $\mathbb{Z}^\bullet X$ , is a variation on the Bousfield-Kan  $\mathbb{Z}$ -resolution of  $X$ .

*Proof.* That  $Q^n\mathcal{C}(X) = \mathcal{C}(\tilde{\mathbb{Z}}^{n+1}X)$  follows by induction and corollary 3.11, which also implies the statements regarding coface and codegeneracies. The final statement follows from example 4.1 of [5, chapter VII, section 4].  $\square$

**Definition 3.16.** Define the functor,  $\mathcal{H}: \mathcal{S}_{\text{cell}} \rightarrow \mathbf{S}_0$ , the category of pointed, reduced simplicial sets by

$$\mathcal{H}C = \text{Tot}(\hat{Q}C) = \text{hom}_{\text{cos}}(\Delta^\bullet, \hat{Q}C)$$

—where  $\text{Tot}(\ast)$  is the total-space functor (see [5, chapter VIII] or [1]) and  $\Delta^\bullet$  is the standard cosimplex. We call  $\mathcal{H}C$  the *Hurewicz realization* of  $C$ .

**Proposition 3.17.** *If  $C \in \mathcal{S}_{\text{cell}}$ , then  $\mathcal{H}C$  is a Kan space, and any surjection  $f: C \rightarrow D$  in  $\mathcal{S}_{\text{cell}}$  induces a fibration*

$$\mathcal{H}f: \mathcal{H}C \rightarrow \mathcal{H}D$$

*If  $f$  is also a homology equivalence, then  $\mathcal{H}f$  is a trivial fibration.*

*Proof.* All of the coface maps except for the 0<sup>th</sup> in  $\hat{Q}C$  are morphisms of simplicial abelian groups. It follows that  $\hat{Q}C$  is “group-like” in the sense of section 4 in chapter X of [1]. The conclusion follows from proposition 4.9 section 4 in chapter X of [1].

If  $f$  is also a homology equivalence, then  $\hat{Q}f: \hat{Q}C \rightarrow \hat{Q}D$  is a pointwise trivial fibration. The final statement follows from theorem 4.13 in chapter VIII of [1], and the fact that  $\Delta^\bullet$  is cofibrant in  $\text{coS}$ .  $\square$

When  $C$  is topologically realizable, we can say a bit more:

**Proposition 3.18.** *If  $X$  is a pointed, reduced simplicial set there exists a natural canonical map*

$$h_X: X \rightarrow \mathcal{HC}(X)$$

*natural with respect to maps of simplicial sets. If  $Y$  is another pointed, reduced simplicial set and*

$$f: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$$

*is a morphism of  $\mathfrak{S}$ -coalgebras, the diagram*

$$(3.5) \quad \begin{array}{ccc} \mathcal{C}(X) & \xrightarrow{f} & \mathcal{C}(Y) \\ \mathcal{C}(h_X) \downarrow & & \downarrow \mathcal{C}(h_Y) \\ \mathcal{C}(\mathcal{HC}(X)) & \xrightarrow{\mathcal{C}(\mathcal{H}f)} & \mathcal{C}(\mathcal{HC}(Y)) \end{array}$$

*Proof.* The  $\mathbb{Z}$ -completion functor comes equipped with a natural augmentation map

$$\phi_X: X \rightarrow \mathbb{Z}^\bullet X$$

Since lemma 3.15 implies that  $Q^\bullet X = \mathcal{C}(\mathbb{Z}^\bullet X)$ , the induces a canonical map

$$\mathcal{C}(\phi_X): \mathcal{C}(X) \rightarrow \mathcal{C}(\mathbb{Z}^\bullet X) = \hat{Q}\mathcal{C}(X)$$

and, applying the  $\text{hom}(\star, *)$  functor gives

$$\text{hom}(\star, \mathcal{C}(\phi_X)): \text{hom}(\star, \mathcal{C}(X)) \rightarrow \text{hom}(\star, \mathcal{C}(\mathbb{Z}^\bullet X))$$

and we define  $h_X$  to be *induced* by the composite (i.e. the morphism of total spaces induced by this)

$$X \xrightarrow{u_X} \text{hom}(\star, \mathcal{C}(X)) \xrightarrow{\text{hom}(\star, \mathcal{C}(\phi_X))} \text{hom}(\star, \mathcal{C}(\mathbb{Z}^\bullet X))$$

Commutativity of diagram 3.5 follows from proposition 3.4, which shows that the diagram

$$\begin{array}{ccc} \mathcal{C}(X) & \xrightarrow{f} & \mathcal{C}(Y) \\ \downarrow & & \downarrow \\ \mathcal{C}(\mathbb{Z}X) & \xrightarrow{L_{\text{cell}} f} & \mathcal{C}(\mathbb{Z}Y) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

of cosimplicial cellular coalgebras commutes at the lowest level.  $\square$

If  $X$  is a pointed, reduced simplicial set, lemma 3.15 implies  $\mathcal{H}\mathcal{C}(X)$  is just  $\mathbb{Z}^\bullet X$  with its outer copy of  $\mathbb{Z}$  replaced by  $\tilde{\Gamma}^*$ . We immediately conclude:

**Corollary 3.19.** *Let  $X$  be a pointed, reduced simplicial set. Then there exists a natural pointwise trivial fibration of cosimplicial spaces*

$$\hat{\gamma}_X: \hat{Q}\mathcal{C}(X) \rightarrow \mathbb{Z}^\bullet X$$

with a pointwise natural trivial cofibration

$$\hat{i}_X: \mathbb{Z}^\bullet X \rightarrow \hat{Q}\mathcal{C}(X)$$

where  $\mathbb{Z}^\bullet X$  is the cosimplicial  $\mathbb{Z}$ -resolution of  $X$  defined in example 4.1 of [5, chapter VII, section 4].

Consequently,

- (1) there exists a natural trivial fibration

$$\bar{\gamma}_X: \mathcal{H}\mathcal{C}(X) \rightarrow \mathbb{Z}_\infty X$$

that makes the diagram

$$(3.6) \quad \begin{array}{ccc} X & \xrightarrow{h_X} & \mathcal{H}\mathcal{C}(X) \\ & \searrow \phi_X & \downarrow \bar{\gamma}_X \\ & & \mathbb{Z}_\infty X \end{array}$$

commute, where  $\phi_X: X \rightarrow \mathbb{Z}_\infty X$  is the canonical map (see 4.2 in [1, chapter I]).

- (2) there exists a natural right inverse  $\bar{i}_X: \mathbb{Z}_\infty X \rightarrow \mathcal{H}\mathcal{C}(X)$  to  $\bar{\gamma}_X$  that is a weak equivalence.

It follows that  $h_X: X \rightarrow \mathcal{H}\mathcal{C}(X)$  is an integral homology equivalence if  $X$  is  $\mathbb{Z}$ -good.

*Remark.* The integral homology equivalence in statement 2 is not necessarily a weak equivalence of cellular coalgebras. Later papers in this series will take up the question of the model structure on the category of cellular coalgebras.

*Proof.* Simply define  $\hat{\gamma}_X$  and  $\hat{\iota}_X$  to be  $\gamma_X$  and  $\iota_X$ , respectively, in each level of  $\hat{Q}\mathcal{C}(X)$  — see lemma 3.6. The maps

$$\hat{\gamma}_X: \hat{Q}\mathcal{C}(X) \rightarrow \mathbb{Z}^\bullet X$$

are all surjective morphisms of group-like cosimplicial spaces and weak equivalences, so they are *trivial fibrations*. They induce a trivial fibration of total spaces  $\bar{\gamma}_X: \mathcal{H}\mathcal{C}(X) \rightarrow \mathbb{Z}_\infty X$  and the induced map  $\bar{\iota}_X: \mathbb{Z}_\infty X \rightarrow \mathcal{H}\mathcal{C}(X)$  is an inverse, hence also a weak equivalence.

The commutativity of diagram 3.6 immediately follows from the way the map  $h_X$  was defined in proposition 3.18.  $\square$

**Corollary 3.20.** *If  $X$  and  $Y$  are pointed, reduced simplicial sets, and*

$$f: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$$

*is a morphism of  $\mathfrak{S}$ -coalgebras,  $f$  induces a map*

$$\mathcal{H}f': \mathbb{Z}_\infty X \rightarrow \mathbb{Z}_\infty Y$$

*that makes the diagram*

$$\begin{array}{ccccc} \mathcal{C}(X) & \xrightarrow{f} & \mathcal{C}(Y) & & \\ \downarrow \mathcal{C}(\phi_X) & \searrow \mathcal{C}(h_X) & & \swarrow \mathcal{C}(h_Y) & \downarrow \mathcal{C}(\phi_Y) \\ & \mathcal{C}(\mathcal{H}\mathcal{C}(X)) & \xrightarrow{\mathcal{C}(\mathcal{H}f)} & \mathcal{C}(\mathcal{H}\mathcal{C}(Y)) & \\ \downarrow \mathcal{C}(\bar{\gamma}_X) & \nearrow \mathcal{C}(\bar{\gamma}_X) & & \swarrow \mathcal{C}(\bar{\gamma}_Y) & \downarrow \mathcal{C}(\bar{\gamma}_Y) \\ \mathcal{C}(\mathbb{Z}_\infty X) & \xrightarrow{\mathcal{C}(\mathcal{H}f')} & & & \mathcal{C}(\mathbb{Z}_\infty Y) \end{array}$$

*commute, where*

$$\phi_X: X \rightarrow \mathbb{Z}_\infty X$$

*is the canonical map (see 4.2 in [1, chapter I]).*

*If  $f$  is surjective and a homology equivalence, then  $\mathcal{H}f$  and  $\mathcal{H}f'$  are weak equivalences.*

*Remark.* If we remove one level of  $\mathcal{C}(\ast)$ , we get a commutative diagram of spaces:

$$\begin{array}{ccccc}
 X & & & & Y \\
 \downarrow \phi_X & \searrow h_X & & \swarrow h_Y & \downarrow \phi_Y \\
 & \mathcal{H}\mathcal{C}(X) & \xrightarrow{\mathcal{H}f} & \mathcal{H}\mathcal{C}(Y) & \\
 \bar{\gamma}_X \swarrow & & & \searrow \bar{\gamma}_Y & \\
 \mathbb{Z}_\infty X & \xrightarrow{\mathcal{H}f'} & & & \mathbb{Z}_\infty Y
 \end{array}$$

*Proof.* The map  $\mathcal{H}f'$  is the composite

$$\mathbb{Z}_\infty X \xrightarrow{\bar{\iota}_X} \mathcal{H}\mathcal{C}(X) \xrightarrow{\mathcal{H}f} \mathcal{H}\mathcal{C}(Y) \xrightarrow{\bar{\gamma}_Y} \mathbb{Z}_\infty Y$$

The upper quadrilateral commutes by proposition 3.18. The left and right triangles commute due to diagram 3.6. The bottom quadrilateral commutes because  $\bar{\gamma}_X \circ \bar{\iota}_X = 1$ , which implies that the outer square also commutes. The final statement follows from corollary 3.19.  $\square$

If spaces are *nilpotent*, we can say a bit more:

**Corollary 3.21.** *Under the hypotheses of corollary 3.20, if  $Y$  is also nilpotent then  $\phi_Y: Y \rightarrow \mathbb{Z}_\infty Y$  is a weak equivalence with a homotopy-inverse,  $\phi'$ , that fits into a commutative diagram*

$$\begin{array}{ccccc}
 X & & & & Y \\
 \downarrow \phi_X & \searrow h_X & & \swarrow h_Y & \downarrow \phi_Y \\
 & \mathcal{H}\mathcal{C}(X) & \xrightarrow{\mathcal{H}f} & \mathcal{H}\mathcal{C}(Y) & \\
 \bar{\gamma}_X \swarrow & & & \searrow \bar{\gamma}_Y & \\
 \mathbb{Z}_\infty X & \xrightarrow{\mathcal{H}f'} & & & \mathbb{Z}_\infty Y
 \end{array}$$

$\phi'$  is indicated by a curved arrow from  $\phi_Y$  back to  $\mathbb{Z}_\infty Y$ .

where

- (1)  $h_Y: Y \rightarrow \mathcal{H}\mathcal{C}(Y)$  is a weak equivalence
- (2) a morphism of cellular coalgebras,  $f: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ , induces a map of simplicial sets

$$X \xrightarrow{\phi_X} \mathbb{Z}_\infty X \xrightarrow{\mathcal{H}f'} \mathbb{Z}_\infty Y \xrightarrow{\phi'} Y$$

*Proof.* The main statement (that  $\phi_Y$  is a weak equivalence) follows from proposition 3.5 in chapter V of [1].  $\square$

**Corollary 3.22.** *If  $X$  and  $Y$  are pointed nilpotent reduced simplicial sets, then  $X$  is weakly equivalent to  $Y$  if and only if there exists a morphism of  $\mathfrak{S}$ -coalgebras*

$$f: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$$

*that is an integral homology equivalence.*

*Proof.* Clearly, if

$$g: X \rightarrow Y$$

is a weak equivalence, then

$$\mathcal{C}(g): \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$$

is a morphism of cellular coalgebras that is an integral homology equivalence.

Conversely, diagram 3.6 in corollary 3.19 implies that

$$\begin{array}{ccc} \mathcal{C}(X) & \xrightarrow{\mathcal{C}(h_X)} & \mathcal{C}(\mathcal{HC}(X)) \\ \mathcal{C}(Y) & \xrightarrow[\mathcal{C}(h_Y)]{} & \mathcal{C}(\mathcal{HC}(Y)) \end{array}$$

are integral homology equivalences. It follows that the homology equivalence

$$f: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$$

induces a homology equivalence

$$(3.7) \quad \mathcal{C}(\mathcal{H}f): \mathcal{C}(\mathcal{HC}(X)) \rightarrow \mathcal{C}(\mathcal{HC}(Y))$$

Now corollary 3.21 implies the existence of weak equivalences

$$\begin{array}{ccc} X & \xrightarrow{h_X} & \mathcal{HC}(X) \\ Y & \xrightarrow[h_Y]{} & \mathcal{HC}(Y) \end{array}$$

so that  $\mathcal{HC}(X)$  and  $\mathcal{HC}(Y)$  are both nilpotent. It follows that integral homology equivalence in 3.7 defines a weak equivalence:

$$\mathcal{H}f: \mathcal{HC}(X) \rightarrow \mathcal{HC}(Y)$$

The conclusion follows.  $\square$

## APPENDIX A. PROOF OF THEOREM 3.10

We begin with a general result:

**Lemma A.1.** *Let  $C$  be a free abelian group on a countable basis, let*

$$\hat{C} = \mathbb{Z} \oplus \prod_{i=1}^{\infty} C^{\otimes i}$$

*Let  $e: C \rightarrow \hat{C}$  be the function that sends  $c \in C$  to*

$$(1, c, c \otimes c, c \otimes c \otimes c, \dots) \in \hat{C}$$

*For any integer  $t > 1$  and any set  $\{c_1, \dots, c_t\} \in C$  of distinct, nonzero elements, the elements*

$$\{e(c_1), \dots, e(c_t)\} \in \mathbb{Q} \otimes_{\mathbb{Z}} \hat{C}$$

*are linearly independent over  $\mathbb{Q}$ . It follows that  $e$  defines an injective function*

$$\bar{e}: \mathbb{Z}[C] \rightarrow \hat{C}$$

*Remark.* Although this seems intuitively clear, it was surprisingly difficult to prove.

*Proof.* We will construct a vector-space morphism

$$(A.1) \quad f: \mathbb{Q} \otimes_{\mathbb{Z}} \hat{C} \rightarrow V$$

such that the images,  $\{f(e(c_i))\}$ , are linearly independent. We begin with the “truncation morphism”

$$r_t: \hat{C} \rightarrow \mathbb{Z} \oplus \bigoplus_{i=1}^{t-1} C^{\otimes i} = \hat{C}_{t-1}$$

which maps  $C^{\otimes 1}$  isomorphically. If  $\{b_i\}$  is a  $\mathbb{Z}$ -basis for  $C$ , we define a vector-space morphism

$$g: \hat{C}_{t-1} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}[X_1, X_2, \dots]$$

by setting

$$g(c) = \sum_{\alpha} z_{\alpha} X_{\alpha}$$

where  $c = \sum_{\alpha} z_{\alpha} b_{\alpha} \in C \otimes_{\mathbb{Z}} \mathbb{Q}$ , and extend this to  $\hat{C}_{t-1} \otimes_{\mathbb{Z}} \mathbb{Q}$  via

$$g(c_1 \otimes \dots \otimes c_j) = g(c_1) \cdots g(c_j) \in \mathbb{Q}[X_1, X_2, \dots]$$

The map in equation A.1 is just the composite

$$\hat{C} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{r_{t-1} \otimes 1} \hat{C}_{t-1} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{g} \mathbb{Q}[X_1, X_2, \dots]$$

It is not hard to see that

$$p_i = f(e(c_i)) = 1 + f(c_i) + \dots + f(c_i)^{t-1} \in \mathbb{Q}[X_1, X_2, \dots]$$

for  $i = 1, \dots, t$ . Since the  $f(c_i)$  are *linear* in the indeterminates  $X_i$ , the degree- $j$  component (in the indeterminates) of  $f(e(c_i))$  is precisely  $f(c_i)^j$ . It follows that a linear dependence-relation

$$\sum_{i=1}^t \alpha_i \cdot p_i = 0$$

with  $\alpha_i \in \mathbb{Q}$ , holds if and only if

$$\sum_{i=1}^t \alpha_i \cdot f(c_i)^j = 0$$

for all  $j = 0, \dots, t-1$ . This is equivalent to  $\det M = 0$ , where

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ f(c_1) & f(c_2) & \cdots & f(c_t) \\ \vdots & \vdots & \ddots & \vdots \\ f(c_1)^{t-1} & f(c_2)^{t-1} & \cdots & f(c_t)^{t-1} \end{bmatrix}$$

Since  $M$  is the transpose of the Vandermonde matrix, we get

$$\det M = \prod_{1 \leq i < j \leq t} (f(c_i) - f(c_j))$$

Since  $f|C \otimes_{\mathbb{Z}} \mathbb{Q} \subset \hat{C} \otimes_{\mathbb{Z}} \mathbb{Q}$  is *injective*, it follows that this *only* vanishes if there exist  $i$  and  $j$  with  $i \neq j$  and  $c_i = c_j$ . The second conclusion follows.  $\square$

**Proposition A.2.** *Let  $X$  be a simplicial set with  $C = \mathcal{C}(X)$  and with coalgebra structure*

$$\Delta_n: RS_n \otimes \mathcal{C}(X) \rightarrow \mathcal{C}(X)^{\otimes n}$$

*and suppose  $RS_2$  is generated in dimension  $n$  by  $e_n = \underbrace{[(1, 2)| \cdots |(1, 2)]}_{n \text{ terms}}$ . If  $x \in C$  is the image of a  $k$ -simplex, then*

$$\Delta(e_k \otimes x) = \epsilon_k \cdot x \otimes x$$

*where  $\epsilon_k = (-1)^{k(k-1)/2}$ .*

**Remark.** This is just a chain-level statement that the Steenrod operation  $Sq^0$  acts trivially on mod-2 cohomology. A weaker form of this result appeared in [4].

*Proof.* For all  $k$ , let  $\Delta^k$  denote the standard  $k$ -simplex, whose vertices are  $\{[0], \dots, [k]\}$  and whose  $j$ -faces are  $\{[i_0, \dots, i_j]\}$ , with

$i_1 < \dots < i_j$ ,  $j \leq k$ . Let  $C$  be the normalized chain-complex of  $\Delta^k$  with augmentation

$$\epsilon: C \rightarrow \mathbb{Z}$$

that maps all 0-simplices to 1 and all others to 0. Recall that  $(RS_2)_n = \mathbb{Z}[\mathbb{Z}_2]$  generated by  $e_n = \underbrace{[(1, 2) | \dots | (1, 2)]}_{n \text{ factors}}$ . Let  $T$  be the generator of  $\mathbb{Z}_2$  — acting on  $C \otimes C$  by swapping the copies of  $C$ .

We assume that  $f(e_i \otimes C(\Delta^j)) \subset C(\Delta^j) \otimes C(\Delta^j)$  so that

$$(A.2) \quad i > j \implies f(e_i \otimes C(\Delta^j)) = 0$$

Define the contracting homotopy:

$$(A.3) \quad \varphi([i_0, \dots, i_t]) = \begin{cases} (-1)^k [i_0, \dots, i_t, k] & \text{if } i_t \neq k \\ 0 & \text{if } i_t = k \end{cases}$$

It is easy to verify that

$$\varphi \circ d + d \circ \varphi = 1 - i \circ \epsilon$$

where  $i: \mathbb{Z} \rightarrow C_0$  maps 1 to  $[k]$ . We extend this to  $C \otimes C$  via

$$\varphi_2 = \varphi \otimes 1 + i \circ \epsilon \otimes \varphi$$

and use the Koszul convention on signs. Note that  $\varphi_2^2 = 0$ . As in section 4 of [10], if  $e_0 \in RS_2$  is the 0-dimensional generator, we define

$$f: RS_2 \otimes C \rightarrow C \otimes C$$

inductively by

$$(A.4) \quad \begin{aligned} f(e_0 \otimes [i]) &= [i] \otimes [i] \\ f(e_0 \otimes [0, \dots, k]) &= \sum_{i=0}^k [0, \dots, i] \otimes [i, \dots, k] \end{aligned}$$

Let  $\sigma = \Delta^k$  and inductively define

$$\begin{aligned} f(e_k \otimes \sigma) &= \varphi_2(f(\partial e_k \otimes \sigma) + (-1)^k \varphi_2 f(e_k \otimes \partial \sigma)) \\ &= \varphi_2(f(\partial e_k \otimes \sigma)) \end{aligned}$$

because of equation A.2.

Expanding  $\varphi_2$ , we get

$$(A.5) \quad \begin{aligned} f(e_k \otimes \sigma) &= (\varphi \otimes 1)(f(\partial e_k \otimes \sigma)) + (i \circ \epsilon \otimes \varphi)f(\partial e_k \otimes \sigma) \\ &= (\varphi \otimes 1)(f(\partial e_k \otimes \sigma)) \end{aligned}$$

because  $\varphi^2 = 0$  and  $\varphi \circ i \circ \epsilon = 0$ .

Noting that  $\partial e_k = (1 + (-1)^k T)e_{k-1} \in RS_2$ , we get

$$\begin{aligned} f(e_k \otimes \sigma) &= (\varphi \otimes 1)(f(e_{k-1} \otimes \sigma)) + (-1)^k(\varphi \otimes 1) \cdot T \cdot f(e_{k-1} \otimes \sigma) \\ &= (-1)^k(\varphi \otimes 1) \cdot T \cdot f(e_{k-1} \otimes \sigma) \end{aligned}$$

again, because  $\varphi^2 = 0$  and  $\varphi \circ i \circ \epsilon = 0$ . We continue, using equation A.5 to compute  $f(e_{k-1} \otimes \sigma)$ :

$$\begin{aligned} f(e_k \otimes \sigma) &= (-1)^k(\varphi \otimes 1) \cdot T \cdot f(e_{k-1} \otimes \sigma) \\ &= (-1)^k(\varphi \otimes 1) \cdot T \cdot (\varphi \otimes 1) \left( f(\partial e_{k-1} \otimes \sigma) \right. \\ &\quad \left. + (-1)^{k-1}f(e_{k-1} \otimes \partial\sigma) \right) \\ &= (-1)^k\varphi \otimes \varphi \cdot T \cdot \left( f(\partial e_{k-1} \otimes \sigma) \right. \\ &\quad \left. + (-1)^{k-1}f(e_{k-1} \otimes \partial\sigma) \right) \end{aligned}$$

If  $k - 1 = 0$ , then the left term vanishes. If  $k - 1 = 1$  so  $\partial e_{k-1}$  is 0-dimensional then equation A.4 gives  $f(\partial e_1 \otimes \sigma)$  and this vanishes when plugged into  $\varphi \otimes \varphi$ . If  $k - 1 > 1$ , then  $f(\partial e_{k-1} \otimes \sigma)$  is in the image of  $\varphi_2$ , so it vanishes when plugged into  $\varphi \otimes \varphi$ .

In all cases, we can write

$$\begin{aligned} f(e_k \otimes \sigma) &= (-1)^k\varphi \otimes \varphi \cdot T \cdot (-1)^{k-1}f(e_{k-1} \otimes \partial\sigma) \\ &= -\varphi \otimes \varphi \cdot T \cdot f(e_{k-1} \otimes \partial\sigma) \end{aligned}$$

If  $f(e_{k-1} \otimes \Delta^{k-1}) = \epsilon_{k-1}\Delta \otimes \Delta^{k-1}$  (the inductive hypothesis), then

$$\begin{aligned} f(e_{k-1} \otimes \partial\sigma) &= \\ &\sum_{i=0}^k \epsilon_{k-1} \cdot (-1)^i [0, \dots, i-1, i+1, \dots, k] \otimes [0, \dots, i-1, i+1, \dots, k] \end{aligned}$$

and the only term that does not get annihilated by  $\varphi \otimes \varphi$  is

$$(-1)^k [0, \dots, k-1] \otimes [0, \dots, k-1]$$

(see equation A.3). We get

$$\begin{aligned}
f(e_k \otimes \sigma) &= \epsilon_{k-1} \cdot \varphi \otimes \varphi \cdot T \cdot (-1)^{k-1}[0, \dots, k-1] \otimes [0, \dots, k-1] \\
&= \epsilon_{k-1} \cdot \varphi \otimes \varphi(-1)^{(k-1)^2+k-1}[0, \dots, k-1] \otimes [0, \dots, k-1] \\
&= \epsilon_{k-1} \cdot (-1)^{(k-1)^2+2(k-1)}\varphi[0, \dots, k-1] \otimes \varphi[0, \dots, k-1] \\
&= \epsilon_{k-1} \cdot (-1)^{k-1}[0, \dots, k] \otimes [0, \dots, k] \\
&= \epsilon_k \cdot [0, \dots, k] \otimes [0, \dots, k]
\end{aligned}$$

where the sign-changes are due to the Koszul Convention. We conclude that  $\epsilon_k = (-1)^{k-1}\epsilon_{k-1}$ .  $\square$

**Lemma A.3.** *Let  $X$  be a pointed reduced simplicial set with  $D = [\mathcal{C}(X)] = C(X)$  and let*

$$c = p_D \circ [\mathcal{C}(\iota_X)]: [\mathcal{C}(\tilde{\mathbb{Z}}X)] \rightarrow D$$

*If  $S \subset [\mathcal{C}(\tilde{\mathbb{Z}}X)_m]$  is the set of elements generated by  $m$ -simplices of  $\tilde{\mathbb{Z}}X$ , then  $c|S: S \rightarrow D_m$  is a bijection.*

*Remark.* The map,  $c$ , is far from injective — we are only making a statement about its restriction to the set of simplicial generators.

*Proof.* This is a consequence of how the map  $p_C: [\mathcal{C}(\tilde{\Gamma}D)] \rightarrow D$  is defined: An  $n$ -simplex  $\sigma \in \tilde{\Gamma}D$  corresponds to a chain map

$$\bar{\sigma}: C(\Delta^n) \rightarrow D$$

and  $p_D([\sigma]) = \bar{\sigma}(C(\Delta^n)_n)$ . Two simplices  $\sigma_1, \sigma_2 \in \tilde{\mathbb{Z}}X$  map to the same element of  $D$  if and only if they define the same element of  $D$ .  $\square$

This leads to the proof:

**Corollary A.4.** *If  $X$  is a pointed, reduced simplicial set, the classifying map*

$$a_{\mathcal{C}(\tilde{\mathbb{Z}}X)}: \mathcal{C}(\tilde{\mathbb{Z}}X) \rightarrow P_{\mathfrak{S}}D$$

*is injective, where  $D = [\mathcal{C}(X)]$ .*

*Proof.* Let  $E = [\mathcal{C}(\tilde{\mathbb{Z}}X)]$ , and let

$$\alpha: \mathcal{C}(\tilde{\mathbb{Z}}X) \rightarrow P_{\mathfrak{S}}E$$

be the coalgebra's classifying map. The results of [11] imply that

$$P_{\mathfrak{S}}E \subset \prod_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(RS_n, E^{\otimes n})$$

and the map  $a_{\mathcal{C}(X)}$  induces a commutative diagram

$$(A.6) \quad \begin{array}{ccccc} \mathcal{C}(\tilde{\mathbb{Z}}X) & \xrightarrow{\alpha} & P_{\mathfrak{S}} E & \longrightarrow & \prod_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(RS_n, E^{\otimes n}) \\ & \searrow a_{\mathcal{C}(\tilde{\mathbb{Z}}X)} & \downarrow P_{\mathfrak{S}} c & & \downarrow \prod_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(1, c^{\otimes n}) \\ & & P_{\mathfrak{S}} D & \longrightarrow & \prod_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(RS_n, D^{\otimes n}) \end{array}$$

where we follow the convention that  $\text{Hom}_{\mathbb{Z}S_0}(RS_0, E^0) = \mathbb{Z}$ ,  $\text{Hom}_{\mathbb{Z}S_1}(RS_1, E) = E$ , and  $c: E \rightarrow D$  is defined in lemma A.3. Let  $p_n$  be projection to a factor

$$p_n: \prod_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(RS_n, E^{\otimes n}) \rightarrow \text{Hom}_{\mathbb{Z}S_n}(RS_n, E^{\otimes n})$$

If  $\sigma \in \tilde{\mathbb{Z}}X$  is an  $m$ -simplex defining an element  $[\sigma] \in E_m$ , proposition A.2 implies that

$$p_2 \circ \alpha([\sigma]) = \epsilon \cdot (e_m \mapsto [\sigma] \otimes [\sigma]) \in \text{Hom}_{\mathbb{Z}S_2}(RS_2, E \otimes E)$$

where  $\epsilon = \pm 1$ , depending on the dimension of  $\sigma$ .

Let  $F_2 = e_m$  and  $F_k = \underbrace{e_m \circ_1 \cdots \circ_1 e_m}_{k-1 \text{ iterations}} \in RS_k$  be the operad-composite (see proposition 2.17 of [10]). The fact that operad-composites map to composites of *coproducts* in a coalgebra implies that

$$p_k \circ \alpha([\sigma]) = \epsilon^{k-1} \cdot (F_k \mapsto \underbrace{[\sigma] \otimes \cdots \otimes [\sigma]}_{k \text{ factors}}) \in \text{Hom}_{\mathbb{Z}S_k}(RS_k, E^{\otimes k})$$

If  $\{\sigma_1, \dots, \sigma_t\} \in \tilde{\mathbb{Z}}X$  are *distinct*  $m$ -simplices representing generators  $\{[\sigma_1], \dots, [\sigma_t]\} \in C(\tilde{\mathbb{Z}}X)_m$  in the chain-complex, lemma A.3 implies that  $\{c[\sigma_1], \dots, c[\sigma_t]\} \in D_m$  are *also* distinct (although no longer *generators*).

Their images in  $\prod_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(RS_n, D^{\otimes n})$  will have the property that

$$p_k \circ a_C([\sigma_i]) = \epsilon^{k-1} \cdot (F_k \mapsto \underbrace{c[\sigma_i] \otimes \cdots \otimes c[\sigma_i]}_{k \text{ factors}}) \in \text{Hom}_{\mathbb{Z}S_k}(RS_k, D^{\otimes k})$$

Evaluation of elements of  $\prod_{n=1}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(RS_n, D^{\otimes n})$  on the sequence  $(\epsilon \cdot E_2, \epsilon^2 \cdot E_3, \epsilon^3 \cdot E_4, \dots)$  gives a homomorphism of  $\mathbb{Z}$ -modules

$$h: \prod_{n=0}^{\infty} \text{Hom}_{\mathbb{Z}S_n}(RS_n, D^{\otimes n}) \rightarrow \prod_{n=0}^{\infty} D^{\otimes n}$$

and  $h \circ a_C(c_i)$  is  $e(c[\sigma_i])$  in lemma A.1 (since  $\epsilon^2 = 1$ ). The conclusion follows from lemma A.1.  $\square$

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